Surfaces and Area.

16.5.1 Find a parameterization of the paraboloid $225z = 9x^2 + 25y^2$, $z \le 4$.

Parameterization of a curve $r(u,v) = f(u,v)\,i + g(u,v)\,j + h(u,v)\,k$

$$z = \frac{9x^2}{225} + \frac{25y^2}{225}$$

$$= \frac{x^2}{\frac{225}{9}} + \frac{y^2}{\frac{225}{25}}$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$ since $z \le 4$.

$u = \theta$ $v = r$ $x = r\cos\theta$ $y = r\sin\theta$

$$\frac{x^2}{25} = 1 \qquad \frac{y^2}{9} = 1$$

$$x^2 = 25 \qquad y^2 = 9$$

$$x = 5 \qquad y = 3$$

$$x = 5\cos\theta \qquad y = 3\sin\theta \quad , \quad 0 \le \theta \le 2\pi$$

since $u = \theta$ and $v = r$

$x = 5v\cos u$ $y = 3v\sin u$

Find z

$$z = \frac{x^2}{25} + \frac{y^2}{9}$$

$$z = \frac{(5v\cos u)^2}{25} + \frac{(3v\sin u)^2}{9}$$

$$z = \frac{\cancel{25}v^2\cos^2 u}{\cancel{25}} + \frac{\cancel{9}v^2\sin^2 u}{\cancel{9}}$$

$$z = v^2\cos^2 u + v^2\sin^2 u$$

$$z = v^2(\cos^2 u + \sin^2 u)$$

$$z = v^2$$

u-limits

since $u = \theta$ and $0 \le \theta \le 2\pi$

then, $0 \le u \le 2\pi$

V-limits

since $0 \le z \le 4$ and $z = v^2$

solving for zero when $z = 0$ $v = 0$ and $z = 4$ $v = 2$

thus, $0 \le v \le 2$.

therefore, $r(u,v) = (5v\cos u)i + (3v\sin u)j + v^2 k$

$0 \le u \le 2\pi$, $0 \le v \le 2$.

16.5.11 Find a parameterization of the portion of the circular cylinder $y^2 + z^2 = 16$ between the planes $x = 0$ and $x = 5$.

the equation is a circle

using polar coordinates $y = r\cos\theta$ $z = r\sin\theta$ $r^2 = 16$

$y^2 + z^2 = 16$ is $y = 4\cos\theta$ $z = 4\sin\theta$, $0 \le \theta \le 2\pi$

thus, $r(u,v) = ui + 4\cos v\, j + 4\sin v\, k$

$0 \le u \le 5$, $0 \le v \le 2\pi$

u-limits let $x = u$

Since x is bounded $x = 0$ and $x = 5$.

$0 \le u \le 5$

v-limits $v = \theta$

$0 \le v \le 2\pi$

16.5.19 use a parameterization to express the area of the surface as a double integral. Then evaluate the integral.

The portion of the cone $z = 4\sqrt{x^2 + y^2}$ between the planes $z = 0$ and $z = 16$.

Let $u = r$ and $v = \theta$ and use cylindrical coordinates to parametize the surface. set up the double integral to find the surface area.

use cylindrical coordinates to parameterize the surface.

$x = r\cos\theta$ $y = r\sin\theta$ $0 \le \theta \le 2\pi$ $v = \theta$ $u = r$

thus,

$x = u\cos v$

$y = u\sin v$

Find z

$z = 4\sqrt{(u\cos v)^2 + (u\sin v)^2}$

$z = 4\sqrt{(u^2\cos^2 v) + (u^2\sin^2 v)}$

$z = 4\sqrt{u^2(\cos^2 v + \sin^2 v)}$

$z = 4\cancel{\sqrt{u^2}}$

$z = 4\sqrt{u^2}$

$z = 4u$

u-limits $u = r$

since $0 \le z \le 16$ and $z = 4u$

Solving for zero when $z = 0$ $u = 0$

and $z = 16$ $u = 4$. thus, $0 \le u \le 4$

v-limits $v = \theta$

$0 \le v \le 2\pi$

thus, $r(u,v) = (u\cos v)i + (u\sin v)j + (4u)k$ $0 \le u \le 4$, $0 \le v \le 2\pi$

Area of $r(u,v) = f(u,v)\,i + g(u,v)\,j + h(u,v)\,k$ is $A = \int_c^d \int_a^b |r_u \times r_v| \, du\, dv$

$r(u,v) = (u\cos v)i + (u\sin v)j + (4u)k$

$r_u = (\cos v)i + (\sin v)j + (4)k$

$r(u,v) = (u\cos v)i + (u\sin v)j + (4u)k$

$r_v = (-u\sin v)i + (u\cos v)j + 0k$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 4 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \begin{vmatrix} \sin v & 4 \\ u\cos v & 0 \end{vmatrix} i - \begin{vmatrix} \cos v & 4 \\ -u\sin v & 0 \end{vmatrix} j + \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} k$$

$$= -4u\cos v - 4u\sin v + (u\cos^2 v - u\sin^2 v)$$
$$= -4u\cos v - 4u\sin v + u$$

$$|r_u \times r_v| = \sqrt{(-4u\cos v)^2 + (-4u\sin v)^2 + u^2}$$
$$= \sqrt{(16u^2\cos^2 v) + (16u^2\sin^2 v) + u^2}$$
$$= \sqrt{16u^2(\cos^2 v + \sin^2 v) + u^2}$$
$$= \sqrt{16u^2 + u^2}$$
$$= \sqrt{17u^2} = \sqrt{17}\,u$$

$$A = \int_c^d \int_a^b |r_u \times r_v| \, du\, dv = \int_0^{2\pi}\int_0^4 \sqrt{17}\,u \; du\, dv = \sqrt{17}\int_0^{2\pi} \left[\frac{u^2}{2}\right]_0^4 dv = \sqrt{17}\int_0^{2\pi} 8\, dv = \sqrt{17}\,[8v]_0^{2\pi} = 16\pi\sqrt{17}$$

16.5.37 Find the area of the surface cut from the paraboloid $x^2 + y^2 + z = 0$ by the plane $z = -2$.

surface area $\displaystyle\int_R\int \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA$

$f(x,y,z) = x^2 + y^2 + z$

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$\nabla f = 2xi + 2yj + 1k$

$|\nabla f| = \sqrt{(2x)^2 + (2y)^2 + 1^2}$

$= \sqrt{4x^2 + 4y^2 + 1}$

$x^2 + y^2 + z = 0$

$x^2 + y^2 - 2 = 0$

$x^2 + y^2 = 2$

The region is the circle $x^2 + y^2 = 2$ parallel to the xy-plane.

$P = 0i + 0j + 1k$

$|\nabla f \cdot P| = |(2xi + 2yj + 1k) \cdot (k)|$

$= 1$

$\displaystyle\int_R\int \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA = \int_R\int \sqrt{4x^2 + 4y^2 + 1}\,/\,1 \; dA$

$\displaystyle = \int\int_{x^2+y^2=2} \sqrt{4x^2 + 4y^2 + 1}\; dA$

since the region of integration is a circle. convert to polar coordinates.

limits

$x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2 + y^2 \quad dA = r\,dr\,d\theta \quad 0 \le r \le \sqrt{2} \quad 0 \le \theta \le 2\pi$

$\displaystyle = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\,dr\,d\theta$

$u = 4r^2 + 1 \quad du = 8r\,dr \quad r\,dr = \frac{1}{8}du$

$r \to 0 \quad u \to 1$

$r \to \sqrt{2} \quad u \to 9$

$\displaystyle = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\,dr\,d\theta$

$\displaystyle = \frac{1}{8}\int_0^{2\pi}\int_1^{9} \sqrt{u}\,du\,d\theta$

$\displaystyle \int \sqrt{u} = \frac{2}{3}u^{\frac{3}{2}} + C$

$\displaystyle = \frac{1}{8}\int_0^{2\pi} \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9 d\theta$

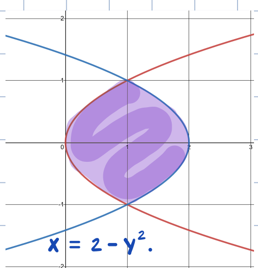
$\displaystyle = \frac{1}{8}\int_0^{2\pi} \left[\frac{2}{3}9^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}}\right] d\theta$

$\displaystyle = \frac{1}{8}\int_0^{2\pi} \frac{52}{3}\, d\theta$

$\displaystyle = \frac{1}{8}\cdot\frac{52}{3}\left[\theta\right]_0^{2\pi} = \frac{13}{6}\,2\pi = \frac{13\pi}{3}$

16.5.39 Find the area of the region cut from the plane $4x + y + 8z = 7$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.

surface area $\displaystyle\int_R\int \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA$



Region of R.

$f(x,y,z) = 4x + y + 8z - 7$

$\nabla f = 4i + 1j + 8k$

$P = k$

$|\nabla f| = \sqrt{4^2 + 1^2 + 8^2}$

$= \sqrt{81} = 9$

$|\nabla f \cdot P| = |(4i + 1j + 8k)\cdot(k)|$

$= 8$

$\displaystyle\int_R\int \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA = \int_R\int \frac{9}{8}\, dA$

$\displaystyle = \frac{9}{8}\int_{-1}^{1}\int_{y^2}^{2-y^2} dx\,dy$

$\displaystyle = \frac{9}{8}\int_{-1}^{1} \left[x\right]_{y^2}^{2-y^2} dy$

$\displaystyle = \frac{9}{8}\int_{-1}^{1} \left[2 - y^2 - y^2\right] dy$

$\displaystyle = \frac{9}{8}\int_{-1}^{1} 2 - 2y^2\, dy$

y-limits (y-intersect)

$x = y^2 \quad x = 2 - y^2$

$y^2 = 2 - y^2$

$2y^2 = 2$

$y^2 = 1$

$y = \pm 1$

$\displaystyle = \frac{9}{8}\left[2y - \frac{2}{3}y^3\right]_{-1}^{1}$

$\displaystyle = \frac{9}{8}\left[2 1 - \frac{2}{3}1^3 - 2 {-1} - \frac{2}{3}{-1}^3\right]$

$\displaystyle = \frac{9}{8}\left[\frac{4}{3} + \frac{4}{3}\right]$

$= 3$

Surface Integrals

16.6.1 Integrate the given function over the given surface.

$G(x,y,z) = z$ over the parabolic cylinder $y = z^2$, $0 \le x \le 2$, $0 \le z \le \frac{\sqrt{3}}{2}$

$r = xi + yj + zk$ $\qquad r_x = 1i$

$= xi + z^2 j + zk$ $\qquad r_z = 2z\,j + 1k$

$$r_x \times r_z = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 2z & 1 \end{vmatrix} = 0i - 1j + 2z\,k = -1j + 2z\,k$$

$$|r_x \times r_z| = \sqrt{(-1)^2 + (2z)^2}$$
$$= \sqrt{1+4z^2}$$

$$\iint_S G(x,y,z)\,d\sigma = \int_0^2 \int_0^{\frac{\sqrt{3}}{2}} z \cdot \sqrt{1+4z^2}\ dz\,dx$$

$u = 1 + 4z^2$ $\quad du = 8z\,dz$ $\quad z\,dz = \frac{1}{8}du$

$v \to 0 \quad u \to 1$

$v \to \frac{\sqrt{3}}{2} \quad u \to 4$

$$= \frac{1}{8} \int_0^2 \int_1^4 \sqrt{u}\ du\,dx$$

$$= \frac{1}{8} \int_0^2 \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^4 dx$$

$$= \frac{1}{8} \int_0^2 \left[\frac{2}{3} 4^{\frac{3}{2}} - \frac{2}{3} 1^{\frac{3}{2}}\right]$$

$$= \frac{1}{8} \int_0^2 \frac{14}{3}\,dx$$

$$= \frac{1}{8}\,\frac{14}{3}\,\left[x\right]_0^2$$

$$= \frac{7}{6}$$

z

16.6.5 Integrate the function $F(x,y,z) = 4z$ over the portion of the plane $x + y + z = 4$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ in the xy-plane.

$x + y + z = 4$

$z = 4 - x - y$

$u = x \qquad v = y \qquad z = 4 - u - v$

$r(u,v) = ui + vj + (4-u-v)k \qquad 0 \le u \le 1,\ 0 \le v \le 1$

$r_u = 1i - 1k$

$r_v = 1j - 1k$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 1i + 1j + 1k$$

$$|r_u \times r_v| = \sqrt{3}$$

$$\iint_S F(x,y,z)\,d\sigma = \int\int 4z \cdot \sqrt{3}\ du\,dv$$

$$= \int_0^1 \int_0^1 4(4-u-v) \cdot \sqrt{3}\ du\,dv$$

$$= \sqrt{3} \int_0^1 \int_0^1 16 - 4u - 4v\ du\,dv$$

$$= \sqrt{3} \int_0^1 \left[16u - 2u^2 - 4uv\right]_0^1 dv$$

$$= \sqrt{3} \int_0^1 14 - 4v\ dv$$

$$= \sqrt{3}\left[14v - \frac{4}{2}v^2\right]_0^1 = 12\sqrt{3}$$

16.6.13 Integrate $g(x,y,z) = x+y+z$ over the portion of the plane $2x+2y+z=8$ that lies in the first octant.

$f(x,y,z) = 2x+2y+z-8$

$\nabla f = 2i + 2j + 1k$

$p = k$

$|\nabla f| = \sqrt{4+4+1}$

$= 3$

$|\nabla f \cdot p| = |(2i+2j+1k)\cdot(k)|$

$= 1$

$2x+2y+z = 8$

$z = 8-2x-2y$

$g(x,y,z) = x+y+z$

$= x+y+(8-2x-2y)$

$= x+y+8-2x-2y$

$= 8-x-y$

Since $2x+2y+z=8$ lies in the first octant

$x=0,\ y=0,\ z=0.$

if $z=0$ then

$2x+2y+z = 8$

$2x+2y+0 = 8$

$2x+2y = 8$

$x+y = 4$ so, $dA = dx\,dy$

x-limits

if $x+y=4$ then

$y = 4-x$

y-limits

Put $y=0$ then

$x = 4-y$

$x = 4-0$

$x = 4$

$$\iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f\cdot p|}\,dA = \iint_R (8-x-y)\cdot\frac{3}{1}\,dA$$

$$= 3\int_{x=0}^{x=4}\int_{y=0}^{y=4-x}(8-x-y)\,dy\,dx$$

$$= 3\int_{x=0}^{x=4}\left[8y-xy-\frac{y^2}{2}\right]_{y=0}^{y=4-x}dx$$

$$= 3\int_{x=0}^{x=4}\left[8(4-x)-x(4-x)-\frac{(4-x)^2}{2}\right]dx$$

$$= 3\int_{x=0}^{x=4}\left(32-8x-4x+x^2-\left(\frac{x^2}{2}-4x+8\right)\right)dx$$

$x^2-2ax+a^2=(x-a)^2$

$$= 3\int_{x=0}^{x=4}\left(32-8x-\cancel{4x}+x^2-\frac{x^2}{2}+\cancel{4x}-8\right)dx$$

$$= 3\int_{x=0}^{x=4}\left(\frac{x^2}{2}-8x+24\right)dx$$

$$= 3\left[\frac{x^3}{6}-4x^2+24x\right]_{x=0}^{x=4}$$

$$= 128$$

16.6.19 use parameterization to find the flux $\iint_S F \cdot n \, d\sigma$ of $F = z^2 i + xj - 2zk$ in the outward direction (normal away from the x-axis) accross the surface cut from the parabolic cylinder $z = 1 - y^2$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Let the parameterization be

$r(x,y) = xi + yj + (1-y^2)k$

$r_x = 1i$

$r_y = 1j - 2yk$

plug $z = 0$ into $z = 1 - y^2$

$0 = 1 - y^2$

$y = \pm 1$ and $0 \le x \le 1$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = (0-0)i - (-2y-0)j + (1-0)k = 2yj + 1k$$

$$|r_x \times r_y| = \sqrt{(2y)^2 + 1^2} = \sqrt{4y^2 + 1}$$

$$F \cdot n \, d\sigma = \left[z^2 i + xj - 2(1-y^2)k\right] \cdot \left[2yj + 1k\right] dy\,dx$$

$$= \left[2xyj - 2(1-y^2)k\right] dy\,dx$$

$$\iint_S F \cdot n \, d\sigma = \iint_R F \cdot (r_x \times r_y) \, dx \, dy = \int_0^1 \int_{-1}^1 (2xyj + 2y^2 - 2) \, dy\,dx$$

$$= \int_0^1 \left[2x\frac{y^2}{2} + \frac{2}{3}y^3 - 2y\right]_{-1}^1 dx$$

$$= \int_0^1 (x + \tfrac{2}{3} - 2) - (x - \tfrac{2}{3} + 2) \, dx$$

$$= \int_0^1 (\cancel{x} + \tfrac{2}{3} - 2 - \cancel{x} + \tfrac{2}{3} - 2) \, dx$$

$$= \int_0^1 (\tfrac{4}{3} - 4) \, dx$$

$$= \left[-\tfrac{8}{3}x\right]_0^1$$

$$= -\tfrac{8}{3}$$

16.6.29 Find the surface integral of the field $F(x,y,z) = -i+4j+4k$ across the rectangular surface $z=0$, $0 \le x \le 3$, $0 \le y \le 2$ in the k direction.

$$Flux = \iint_S F \cdot n \, d\sigma, \quad n = \pm\frac{\nabla g}{|\nabla g|}, \quad d\sigma = \frac{\nabla g}{|\nabla g \cdot p|}$$

$p = k$

$g(x,y,z) = z$

$\nabla g = 0i + 0j + 1k$

$|\nabla g| = \sqrt{1^2}$

$= 1$

$|\nabla g \cdot p| = |(0i + 0j + 1k)\cdot(k)|$

$= 1$

$n = \dfrac{\nabla g}{|\nabla g|} = \pm\dfrac{1k}{1} = \pm k$ choose + direction

$F \cdot n = (-i+4j+4k)\cdot(k)$

$= 4$

$d\sigma = \dfrac{\nabla g}{|\nabla g \cdot p|} = \dfrac{1}{1} = 1$

$$= \int_0^3 \int_0^2 4 \, dy\, dx$$

$$= \int_0^3 \left[4y\right]_0^2 dx$$

$$= \int_0^3 8 \, dx$$

$$= \left[8x\right]_0^3$$

$$= 24$$

Stoke's Theorem.

16.7.1 Find the curl of the vector field $F = (x+y-z)i + (4x-y+2z)j + (2x+9y+z)k$

$M = x+y-z \qquad N = 4x-y+2z \qquad P = 2x+9y+z$

$$curl\ F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)k$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}(x+y-z)$

$= 1$

$\dfrac{\partial M}{\partial z} = \dfrac{\partial}{\partial z}(x+y-z)$

$= -1$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}(4x-y+2z)$

$= 4$

$\dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}(4x-y+2z)$

$= 2$

$\dfrac{\partial P}{\partial x} = \dfrac{\partial}{\partial x}(2x+9y+z)$

$= 2$

$\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}(2x+9y+z)$

$= 9$

$= (9-2)i + (-1-2)j + (4-1)k$

$= 7i - 3j + 3k$

16.7.5 Find the curl of the vector field $F = (x^7yz)i + (xy^2z)j + (xyz^8)k$

$M = x^7yz \qquad N = xy^2z \qquad P = xyz^8$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^7yz) = x^7z$

$\frac{\partial M}{\partial z} = \frac{\partial}{\partial z}(x^7yz) = x^7y$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xy^2z) = y^2z$

$\frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(xy^2z) = xy^2$

$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(xyz^8) = yz^8$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xyz^8) = xz^8$

$$curl\ F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)k$$

$$= (xz^8 - xy^2)i + (x^7y - yz^8)j + (y^2z - x^7z)k$$

$$= x(z^8 - y^2)i + y(x^7 - z^8)j + z(y^2 - x^7)k$$

16.7.7 Use the surface integral in Stokes' Theorem to calculate the circulation of the field $F = x^2i + 5xj + z^2k$ around the curve C: the ellipse $25x^2 + 9y^2 = 10$ in the xy-plane, counterclockwise when viewed from above.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 5x & z^2 \end{vmatrix} = \left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(5x)\right]i - \left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right]j + \left[\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(x^2)\right]k$$

$$= (0-0)i - (0-0)j + (5-0)k$$

$$= 5k$$

$$\oint_C F \cdot dr = \int_S\int \nabla \times F \cdot n\, d\sigma \qquad n = k,\ d\sigma = dx\,dy$$

$$= \int_S\int 5k \cdot k\, dx\, dy$$

$$= \int_R\int 5\, dA = 5\pi ab$$

$$= 5\pi\left(\frac{\sqrt{10}}{5}\right)\left(\frac{\sqrt{10}}{3}\right)$$

$$= 5\pi\left(\frac{10}{15}\right)$$

$$= \frac{10\pi}{3}$$

$y = 0$

$25x^2 = 10$

$x^2 = \frac{10}{25}$

$x^2 = \sqrt{\frac{10}{25}}$

$x = \frac{\sqrt{10}}{5}$

$x = 0$

$9y^2 = 10$

$y^2 = \frac{10}{9}$

$y^2 = \sqrt{\frac{10}{9}}$

$y = \frac{\sqrt{10}}{3}$

16.7.13 Let n be the outer unit normal of the elliptical shell S: $4x^2+4y^2+16z^2=16$, $z \ge 0$ and let $F = yi + x^2 j + (x^2+y^4)^{\frac{3}{2}} \sin e^{\sqrt{xyz}} k$

Find the value of $\int_S \int \nabla \times F \cdot n \, d\sigma$

* $4x^2+4y^2+16z^2=16$, $z \geq 0$

Plug $z=0$, $4x^2+4y^2=16$

$\Rightarrow \boxed{x^2+y^2=4}$

Let $r(t) = \langle 2\cos t, 2\sin t, 0 \rangle$ where $\boxed{0 \le t \le 2\pi}$

$r'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$F = \langle y, x^2, (x^2+y^4)^{3/2} \sin e^{\sqrt{xyz}} \rangle$

$F(r(t)) = \langle 2\sin t, 4\cos^2 t, (4\cos^2 t + 16\sin^4 t)^{3/2} \sin e^0 \rangle$

$\iint_S \nabla F \cdot n ds = \int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt$

$= \int_0^{2\pi} \langle 2\sin t, 4\cos^2 t, (4\cos^2 t + 16\sin^4 t)^{3/2} \sin(1) \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle dt$

$= \int_0^{2\pi} (-4\sin^2 t + 8\cos^3 t + 0) \, dt$

$= \int_0^{2\pi} \left(-4\left(\frac{1-\cos(2t)}{2}\right) + 8\cos^2 t * \cos t\right) dt$

$= \int_0^{2\pi} (-2 + 2\cos(2t) + 8(1-\sin^2 t)\cos t) \, dt$

$= \int_0^{2\pi} (-2 + 2\cos(2t) + 8\cos t - 8\sin^2 t \cos t) \, dt$

$= \left[-2t + \frac{2\sin(2t)}{2} + 8\sin t - \frac{8\sin^3 t}{3}\right] \Big|_0^{2\pi}$

$= (-2*2\pi + 0 + 0 - 0) - (-0 + 0 + 0 - 0)$

$= \boxed{-4\pi}$

16.8.1 Find the divergence of the field $F = (-6x+y+4z)i + (3x-y+7z)j + (3x-y+7z)k$

$$div\ F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

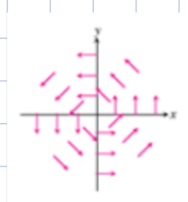
$M = (-6x+y+4z)$ $N = (3x-y+7z)$ $P = (3x-y+7z)$

$div\ F = \frac{\partial}{\partial x}(-6x+y+4z) + \frac{\partial}{\partial y}(3x-y+7z) + \frac{\partial}{\partial z}(3x-y+7z)$

$= -6 - 1 + 7$

$= 0$

16.8.5 Find the divergence of the spin field $F = \frac{-4yi+4xj}{(x^2+y^2)^{1/2}}$



$$div\ F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$M = -\frac{4y}{(x^2+y^2)^{1/2}}$ $N = \frac{4x}{(x^2+y^2)^{1/2}}$ $P = 0$

$= -4y(x^2+y^2)^{-1/2}$ $= 4x(x^2+y^2)^{-1/2}$

$div\ F = \frac{\partial}{\partial x} -4y(x^2+y^2)^{-1/2} + \frac{\partial}{\partial y} 4x(x^2+y^2)^{-1/2}$

$= -4y\left[-\frac{1}{\cancel{2}}(x^2+y^2)^{-\frac{3}{2}} \cdot (\cancel{2}x)\right] + 4x\left[-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot (2y)\right]$

$= \frac{4xy}{(x^2+y^2)^{3/2}} - \frac{4xy}{(x^2+y^2)^{3/2}}$

$= \frac{4xy-4xy}{(x^2+y^2)^{3/2}} = 0$

16.8.9 Use the divergence theorem to find the outward flux of F across the boundary of the region D.

$F = (3y-x)i + (2z-y)j + (4y-4x)k$

D: the cube bounded by the planes $x = \pm 3$, $y = \pm 3$, and $z = \pm 3$.

$M = (3y-x) \qquad N = (2z-y) \qquad P = (4y-4x)$

$$\text{div } F = \frac{\partial}{\partial x}(3y-x) + \frac{\partial}{\partial y}(2z-y) + \frac{\partial}{\partial z}(4y-4x)$$

$$= -1 - 1 + 0$$

$$= -2$$

$$\iint_S F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dV = \iiint_D -2 \, dV$$

$$= \int_{-3}^{3}\int_{-3}^{3}\int_{-3}^{3} -2 \, dx\,dy\,dz = -2\int_{-3}^{3}\int_{-3}^{3} \Big[x\Big]_{-3}^{3} dy\,dz$$

$$= -2\int_{-3}^{3}\int_{-3}^{3} 6\,dy\,dz = -2\int_{-3}^{3}\Big[6y\Big]_{-3}^{3} dz$$

$$= -2\int_{-3}^{3} 36\,dz = -2\Big[36z\Big]_{-3}^{3}$$

$$= -432$$

16.8.11 Use the divergence theorem to find the outward flux of $F = 7yi + 5xyj - 4zk$ across the boundary of the region D: the region inside the solid cylinder $x^2+y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2+y^2$.

$M = 7y \qquad N = 5xy \qquad P = -4z$

$$\text{div } F = \frac{\partial}{\partial x} 7y + \frac{\partial}{\partial y} 5xy + \frac{\partial}{\partial z} -4z$$

$$= 0 + 5x - 4$$

$$= 5x - 4 \qquad\qquad 5x-4$$

$x = r\cos t \quad y = r\sin t \quad r^2 = x^2+y^2 = 4 \rightarrow r = 2 \quad z = r^2$

D: $0 \le r \le 2$, $\quad 0 \le t \le 2\pi$ and $0 \le z \le r^2$

$$\iint_S F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dV = \iiint_D 5x - 4 \, dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{r^2} (5r\cos t - 4)r \, dz\,dr\,dt$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{r^2} 5r^2\cos t - 4r \, dz\,dr\,dt$$

$$= \int_0^{2\pi}\int_0^{2} (5r^2\cos t - 4r)\Big[z\Big]_0^{r^2} dr\,dt$$

$$= \int_0^{2\pi}\int_0^{2} (5r^4\cos t - 4r^3)\,dr\,dt$$

$$= \int_0^{2\pi} \Big[r^5\cos t - r^4\Big]_0^2 dt$$

$$= \int_0^{2\pi} 32\cos t - 16 \, dt$$

$$= \Big[32\sin t - 16t\Big]_0^{2\pi}$$

$$= 32\sin 2\pi - 16\,2\pi - 32\sin 0 - 16\,0$$

$$= 0 - 32\pi - 0 \qquad \sin 2\pi = \sin 0 = 0$$

$$= -32\pi$$